Bellman Equations

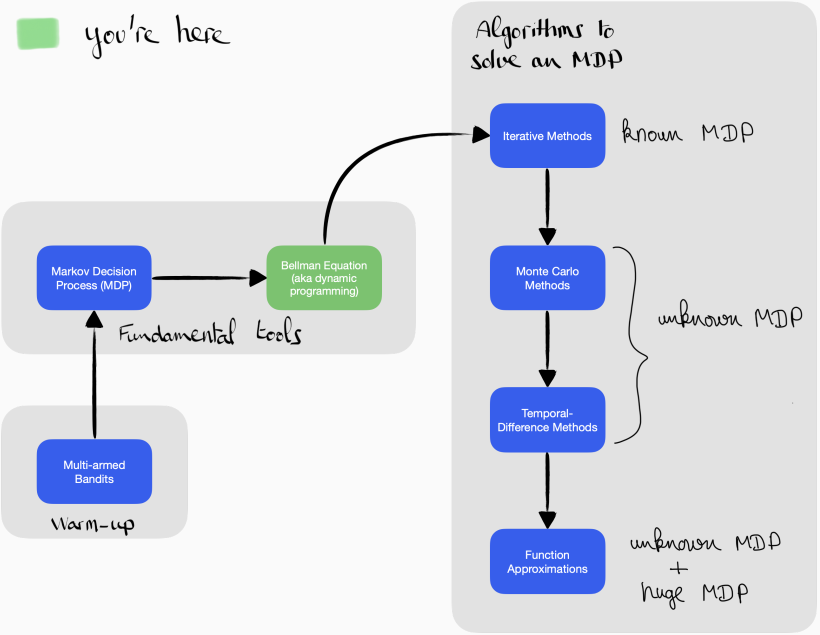
aka Dynamic Programming

Bellman Equations

Using Bellman Equations to Estimate the Value of a Policy

Iterative Methods

Bellman Equations



you're here
Algorithms to solve an MDP
Iterative Methods
known MDP
Markov Decision Process (MDP)
Bellman Equation (aka dynamic programming)
Fundamental tools
Monte Carlo Methods
unknown MDP
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unknown MDP
+
huge MDP

Solving an MDP

Prediction

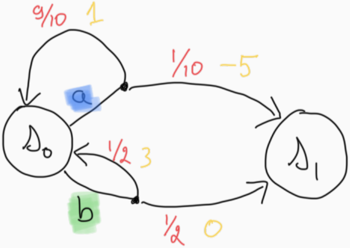
- **Estimate**: $v_\pi(s)$ or $q_\pi(s, a)$ for a given policy π
  Also called: Policy Evaluation
- **Key Question**:
  $>$ Given my strategy, what is my expected return?

Control

- **Estimate**: $\pi_*(s)$ or $q_*(s, a)$
  Goal: Find the Optimal Policy
- **Key Question**:
  $>$ What is the optimal way to behave? For example, what is the best treatment?

Bellman Equations - An Example

Evaluate the value of two different policies for this simple MDP.



$\pi_0(s_0) = a$

$\pi_1(s_0) = b$

- Blackboard.

Bellman Equations for Deterministic Policies

- **State-Value Function**:

$$V^\pi(s) = \sum_{s' \in S, r \in R} p(s', r | s, \pi(s)) \left[r + \gamma V^\pi(s') \right]$$

- **Action-Value Function**:

$$Q^\pi(s, a) = \sum_{s' \in S, r \in R} p(s', r | s, a) \left[r + \gamma V^\pi(s') \right]$$

The equations provide recursive relationships for evaluating a policy.

Richard E. Bellman (1920-1984)

- **State-Value Function**:

$$V^{\pi}(s) = \sum_{s' \in S, r \in R} p(s', r|s, \pi(s)) \left[r + \gamma V^{\pi}(s') \right]$$

Question

What if the policy is probabilistic? $\pi : S \times A \rightarrow [0, 1]$

Optimal Bellman Equations

Same but for V^* and Q^*.

Optimal Bellman Equations

- **State-Value Function**:

$$\forall s \in S, V^*(s) = \max_a \sum_{s' \in S, r \in R} p(s', r | s, a) \left[r + \gamma V^*(s') \right]$$

- **Action-Value Function**:

$$\forall s \in S, a \in A, Q^*(s, a) = \sum_{s' \in S, r \in R} p(s', r | s, a) \left[r + \gamma V^*(s') \right]$$

Again, the equations provide recursive relationships for finding the optimal policy, but this time the system of equations is non-linear.

Unique Solution

Theorem

V^* is the unique solution to the following system of equations:

$$\forall s \in S, \quad V(s) = \max_a \sum_{s' \in S, r \in R} p(s', r|s, a) \left[r + \gamma V(s')\right]$$

All(most) algorithms for solving MDPs are based on Bellman equations in some way.

Using Bellman Equations to Estimate the Value of a Policy

First Idea: Solve the System of Equations

Suppose there are N states. To estimate the value of a policy, write down all equations given by Bellman equations:

$$\begin{cases} V^\pi(s_0) = \sum_{s',r} p(s', r \mid s_0, \pi(s_0)) \left[r + \gamma V^\pi(s')\right] \\ V^\pi(s_1) = \sum_{s',r} p(s', r \mid s_1, \pi(s_1)) \left[r + \gamma V^\pi(s')\right] \\ \quad \vdots \\ V^\pi(s_N) = \sum_{s',r} p(s', r \mid s_N, \pi(s_N)) \left[r + \gamma V^\pi(s')\right] \end{cases}$$

This represents a system of linear equations.

Matrix Form

Let's rewrite the system of equations in matrix form.

$$V^\pi(s) = \sum_{s',r} p(s', r \mid s, \pi(s)) \left[r + \gamma V^\pi(s') \right]$$

can be written as:

$$r(s, \pi(s)) + \gamma \sum_{s'} p(s' \mid s, \pi(s)) V^\pi(s')$$

where:

- $r(s, \pi(s)) = \mathbb{E}[r \mid s = s, a = \pi(s)]$ is the expected immediate reward for taking action $\pi(s)$ in state s.

Thus we get

$$
\begin{cases}
V^\pi(s_0) = r(s_0, \pi(s_0)) + \gamma \sum_{s'} p(s' \mid s_0, \pi(s_0)) V^\pi(s') \\
V^\pi(s_1) = r(s_1, \pi(s_1)) + \gamma \sum_{s'} p(s' \mid s_1, \pi(s_1)) V^\pi(s') \\
\quad \vdots \\
V^\pi(s_N) = r(s_N, \pi(s_N)) + \gamma \sum_{s'} p(s' \mid s_N, \pi(s_N)) V^\pi(s')
\end{cases}
$$

Let $\mathbf{V}^\pi$ and $\mathbf{R}^\pi$ be the vectors of values and rewards, and $\mathbf{P}^\pi$ the transition matrix. Then the system of equations can be written as:

Matrix Representation

$$\mathbf{V}^\pi = \mathbf{R}^\pi + \gamma \mathbf{P}^\pi \mathbf{V}^\pi$$

Thus we get

$$
\begin{cases}
V^\pi(s_0) = r(s_0, \pi(s_0)) + \gamma \sum_{s'} p(s' \mid s_0, \pi(s_0)) V^\pi(s') \\
V^\pi(s_1) = r(s_1, \pi(s_1)) + \gamma \sum_{s'} p(s' \mid s_1, \pi(s_1)) V^\pi(s') \\
\quad \vdots \\
V^\pi(s_N) = r(s_N, \pi(s_N)) + \gamma \sum_{s'} p(s' \mid s_N, \pi(s_N)) V^\pi(s')
\end{cases}
$$

Let $\mathbf{V}^\pi$ and $\mathbf{R}^\pi$ be the vectors of values and rewards, and $\mathbf{P}^\pi$ the transition matrix. Then the system of equations can be written as:

Matrix Representation

$$\mathbf{V}^\pi = \mathbf{R}^\pi + \gamma \mathbf{P}^\pi \mathbf{V}^\pi$$

Solution

$$\mathbf{V}^\pi = (\mathbf{I} - \gamma \mathbf{P}^\pi)^{-1} \mathbf{R}^\pi$$

Computational Complexity

$\mathbf{V}^\pi = (\mathbf{I} - \gamma \mathbf{P}^\pi)^{-1} \mathbf{R}^\pi$

- Size of the matrix: $|S| \times |S|$
- Complexity of matrix inversion:
  - $O(|S|^3)$ using Gauss-Jordan elimination
  - $O(|S|^{2.807})$ using Strassen's algorithm
  - $O(|S|^{2.376})$ using Coppersmith-Winograd algorithm

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Problems

- This is too slow for large MDPs.
- This idea cannot be used for the optimal policy because the system of equations is non-linear :-(

We need to find a **faster** and **more general** algorithm.

Iterative Methods

Fixed-Point computation

We have a function $f : \mathbb{R}^n \to \mathbb{R}^n$ and we want to find a fixed point: a point x^* such that $f(x^*) = x^*$.

Fixed-Point computation

We have a function $f : \mathbb{R}^n \to \mathbb{R}^n$ and we want to find a fixed point: a point x^* such that $f(x^*) = x^*$.

α-contraction

A function is an α-contraction iff

$$\forall x, y \in \mathbb{R}^n, \quad \|f(x) - f(y)\| \leq \alpha \cdot \|x - y\|$$

for some $\alpha \in [0, 1)$.

Banach Theorem

If f is an α-contraction, then

- there exists a unique fixed point x^*
- the sequence $x_{k+1} = f(x_k)$ converges to x^* for any initial point x_0
- it converges exponentially fast: $\|x^* - x_k\| \leq \frac{\alpha^k}{1-\alpha}\|x_1 - x_0\|$

Interlude

Computing the constant $\pi (= 3.141...)$ using Banach's theorem

Interlude

Computing the constant $\pi (= 3.141...)$ using Banach's theorem

$\rightarrow$ Fixed-point of $f(x) = sin(x) + x$

We can use this to solve the Bellman equations

Three main algorithms derived from Banach's theorem:

- **Policy Evaluation.** Find the value $\mathbf{V}^{\pi}$ of a policy π.
- **Value Iteration.** Find the optimal value function $\mathbf{V}^{*}$.
- **Policy Iteration.** Find the optimal policy π^{*}.

Policy Evaluation - Finding the Value $\mathbf{V}^\pi$ of a Policy π

Given a policy π, we define the Bellman operator $T^\pi : \mathbb{R}^{|S|} \rightarrow \mathbb{R}^{|S|}$ as:

$$(T^\pi(\mathbf{V}))(s) = \sum_{s',r} p(s', r \mid s, \pi(s)) \left[r + \gamma \mathbf{V}(s')\right]$$

Finding the value of a policy is equivalent to finding the fixed point of T^π, i.e. $\mathbf{V}^\pi = T^\pi(\mathbf{V}^\pi)$.

It is possible to prove that T^π is a α-contraction, therefore we can apply Banach's theorem to find the fixed point. The algorithm is called **Policy Evaluation**.

Policy Evaluation Algorithm

1. Initialize $\mathbf{V}_0$ randomly
2. Repeat until convergence:
   - For each state s, update $V_{k+1}(s) = \sum_{s',r} p(s', r \mid s, \pi(s)) \left[r + \gamma V_k(s')\right]$

Policy Evaluation Algorithm

1. Initialize $\mathbf{V}_0$ randomly
2. Repeat **until convergence**:
    - For each state s, update $V_{k+1}(s) = \sum_{s',r} p(s', r \mid s, \pi(s)) \left[r + \gamma V_k(s')\right]$

What does **until convergence** mean?

- The difference between two consecutive values is smaller than a threshold: $\| V^{\pi}_{k+1} - V^{\pi}_{k} \| < \epsilon$
- There is a fixed number of iterations: "repeat N times"
- ...

Value Iteration

This is very similar to Policy Iteration. This time, we want to compute $\mathbf{V}^*$, the optimal value function. We define as before the optimal Bellman operator $T^* : \mathbb{R}^{|S|} \to \mathbb{R}^{|S|}$ as:

$$(T^*(\mathbf{V}))(s) = \max_a \sum_{s',r} p(s', r \mid s, a) \left[r + \gamma \mathbf{V}(s')\right]$$

Value Iteration Algorithm

1. Initialize $\mathbf{V}_0$ randomly
2. Repeat **until convergence**:
   - For each state s, update $V_{k+1}(s) = \max_a \sum_{s',r} p(s', r \mid s, a) \left[r + \gamma V_k(s')\right]$

Optimal Policy

Once we have the optimal value function $\mathbf{V}^*$, we can find the optimal policy π^* by taking the greedy policy with respect to $\mathbf{V}^*$:

$$\pi^*(s) = \arg\max_a \sum_{s',r} p(s', r \mid s, a) \left[r + \gamma \mathbf{V}^*(s') \right]$$

We can also apply directly Value Iteration to find the optimal action-value function $\mathbf{Q}^*$, in which case we get the optimal policy directly using:

$$\pi^*(s) = \arg\max_a \mathbf{Q}^*(s, a)$$

Policy Iteration

Value Iteration: $V_0 \to V_1 \to V_2 \to \ldots \to V^*$

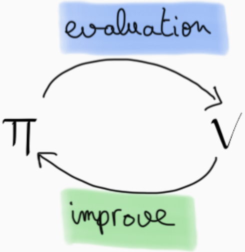
Policy Iteration: $\pi_0 \to \pi_1 \to \pi_2 \to \ldots \to \pi^*$

–> strategy improvement.

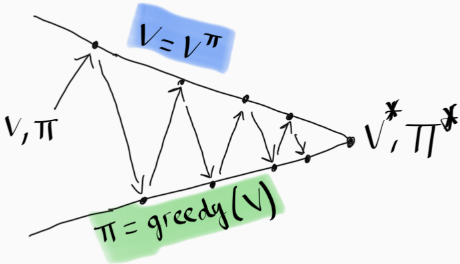
It works in two steps that are repeated until convergence:

$$\pi_k \to V^{\pi_k} \to \pi_{k+1}$$

- **Evaluation.** $\pi_k \to V^{\pi_k}$
  - Compute the value function V^{π_k} for the policy π_k.
  - How? Use policy evaluation.
- **Improvement.** $V^{\pi_k} \to \pi_{k+1}$
  - Define a new policy π_{k+1} that is greedy with respect to Q^{π_k}.
  - $\pi_{k+1}(s) = \arg\max_a Q^{\pi_k}(s, a)$



Policy Iteration's loop



Policy Iteration's convergence

Iterative methods for the Q-value function

Sanity check

Verify that you can apply the same iterative methods to the Q-value function as well.

Complexity

- **Learning the value function** is $O(|S|^2 \cdot |A|)$ per iteration.
- **Learning the q-value function** is $O(|S|^2 \cdot |A|^2)$ per iteration.
- **Playing using the value function** is $O(|S| \cdot |A|)$.
- **Playing using the q-value function** is $O(|A|)$.

Value Iteration vs Policy Iteration

- **Value Iteration** is simpler and often faster.
- **Value Iteration**. Convergence is only asymptotic.
- **Policy Iteration** is more stable and can be more efficient in some cases.
- **Policy Iteration** is guaranteed to converge to the optimal policy in a finite number of steps, while **Value Iteration** converges to the optimal value function but not necessarily in a finite number of steps.
- In **Policy Iteration** we know when to stop: when the policy does not change anymore, it means we have found the optimal policy.
- **Policy Iteration**. More expensive per iteration because it requires policy evaluation.

Summary

- **Bellman Equations** provide recursive relationships for the value of a policy or the optimal value of an MDP.

$$V^{\pi}(s) = \sum_{s',r} p(s', r \mid s, \pi(s)) \left[r + \gamma V^{\pi}(s') \right]$$

$$V^{*}(s) = \max_{a} \sum_{s',r} p(s', r \mid s, a) \left[r + \gamma V^{*}(s') \right]$$

- **Iterative Methods** are used to solve the Bellman equations. Based on fixed-point computation and Banach's theorem.

$$X_{k+1} = f(X_k)$$

- **Policy Evaluation**, **Value Iteration**, and **Policy Iteration** are powerful examples of iterative methods applied to MDPs.