Games and controller synthesis - Reachability games

As a general rule, all game arenas in our exercises are *finite*.

Exercise 1

We reconsider the winning condition $\mathrm{Reach}(F_1) \wedge \mathrm{Reach}(F_2)$ for P_0, where $F_1 \cap F_2 = \emptyset$. So P_0 wins a play if and only if the play visits both F_1 and F_2.

Describe the winning strategies of both players and show that both need memory in order to win.

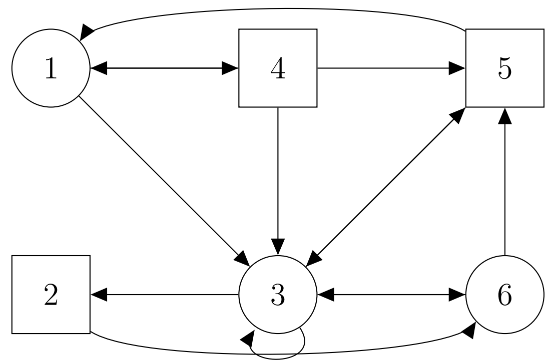
Hint : show that $X = \mathrm{Attr}_0((F_1 \cap \mathrm{Attr}_0(F_2)) \cup (F_2 \cap \mathrm{Attr}_0(F_1))$ is the winning region W_0 of P_0.

Exercise 2

We consider the following winning condition Win for P_0, given two disjoint sets $F, B \subseteq V$.

A play π is won by P_0 if it reaches F without visiting B before. So $\mathrm{Win} = (V \setminus B)^* F V^\omega$.

1. Adapt the attractor construction to this winning condition. What guarantees P_1's winning strategy on the complement of the attractor?
2. Compute the winning regions on the following example, with $F = \{1, 2\}$ and $B = \{5, 6\}$:



Exercise 3

We consider the winning condition Win = Reach(F) $\vee$ Avoid(B) for P_0, where $F \cap B = \emptyset$. A maximal play π is won by P_0 if and only if π visits F or avoids B. In other words, Win $= V^*FV^\omega \cup (V \setminus B)^\omega$.

We want to compute the winning regions W_0, W_1, and winning strategies σ_0, σ_1.

1. What is the winning condition for P_1?
2. Compute the winning regions for this game on the example in Exercise 2 with $F = \{4\}$ and $B = \{5\}$.
3. Give an example of a game with condition Win$' =$ Reach(F)$\wedge$Avoid(B) and a vertex v that is winning for P_0 for condition Reach(F), *and* for condition Avoid(B), but not for Win.
4. Reduce the game with condition Win to a game on a larger arena, where the players remember which sets were visited. What is the winning condition of P_0 for the larger game?
5. Solve the game with condition Win as a *weak parity game*.

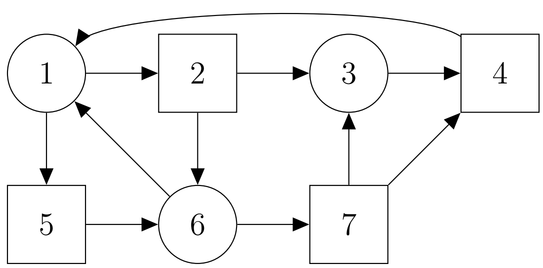
Exercise 4

We consider the winning condition Win $=$ (Reach(F_1) $\wedge$ Avoid(B_1)) $\vee$ (Reach(F_2) $\wedge$ Avoid(B_2)) for P_0, where the sets F_1, F_2, B_1, B_2 are pairwise disjoint. So a maximal play π is won by P_0 if and only if π visits F_1 and avoids B_1, or it visits F_2 and avoids B_2.

Compute the winning regions W_0, W_1, and winning strategies σ_0, σ_1.

Obligation games. Win is a boolean combination of reachability conditions. *Example*: visit either p and q and not r, or q and r and not p. Equivalent formulation:

- Win $= \mathcal{F}$, where $\mathcal{F} = \{F_1, \ldots, F_k\}$, $F_i \subseteq V$.
- A maximal play $\pi = v_0, v_1, \ldots$ is winning (for P_0) if $\mathrm{Occ}(\pi) \in \mathcal{F}$. Notation: $\mathrm{Occ}(\pi) = \{v \mid \exists i \text{ s.t. } v = v_i\}$ is the set of states visited by π.



Win: $\pi \in$ Win if and only if $\{2, 7\} \subseteq \mathrm{Occ}(\pi)$. In order to win, P_0 needs memory in state 1: she has to visit both 2 **and** 5.

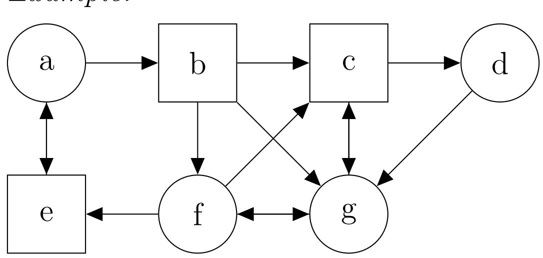
Solution for obligation games: reduction to **weak parity games**.

Weak parity games.

- The game arena $\mathcal{A}$ is equipped with a priority function $p : V \to \{0, \ldots, d\}$: $p(v)$ is the priority (color) of v.
- Win is the set of maximal plays such that the *biggest* priority is even. Formally:

$$\pi \in \text{Win} \quad \text{iff} \quad \max\{p(v) \mid v \in \mathrm{Occ}(\pi)\} \text{ is even}$$

Example.



$p(a) = p(e) = 0$, $p(b) = p(f) = 1$, $p(c) = p(g) = 2$, $p(d) = 3$.

- $d \in W_1$ and $\mathrm{Attr}_1(d) = \{d, c, b\} \subseteq W_1$
- $V' = V \setminus \{b, c, d\}$ is a trap for P_1: player P_0 can keep the game within V'. In V', P_0 can avoid priority 3, and $\mathrm{Attr}_0(g) \mid_{V'} = \{g, f\} \subseteq W_0$
- $V'' = V' \setminus \{f, g\}$ is a trap for P_0: P_1 can keep the game within V''.

  $W_1 \mid_{V''} = \emptyset$, $W_0 \mid_{V''} = \{a, e\}$

Conclusion: $W_0 = \{a, e, f, g\}$, $W_1 = \{b, c, d\}$. Both players have memoryless strategies.

Polyomial-time solution for weak parity games: decompose the game arena into attractors.