Games and synthesis
Regular games

1 Regular games

Game arenas.

$$\mathcal{A} = (V_0, V_1, E)$$

- 2 players: P_0 et P_1. The set of vertices is partitioned as $V = V_0 \cup V_1$, with V_0 belonging to P_0 and V_1 to P_1.
- Edges $E \subseteq V \times V$.

Plays. A play π is a maximal path in the graph (V, E). We usually assume that the graph has no dead-ends.

Strategies.

- A strategy of P_0 is a mapping $\sigma_0 : V^* V_0 \to V$ s.t. $\sigma_0(\pi v) \in post(v)$.

  Here: $\pi\, v$ path in (V, E), $\pi \in V^*$, $v \in V$, and $post(v) = \{w \mid (v, w) \in E\}$. A strategy σ_1 of P_1 is defined similarly.
- A play $\pi = v_0, v_1, \ldots$ is consistent with a strategy σ if $v_{i+1} = \sigma(v_0 \cdots v_i)$ for all i s.t. $v_i \in V_0$.

Game. A game $\mathcal{G}$ consists of an arena $\mathcal{A}$ and a winning condition $Win \subseteq V^\omega$ (winning usually for P_0).

Winner.

- A play π is won by P_0 if $\pi \in Win$.
- $\sigma_0 : V^* V_0 \to V$ is a winning strategy for P_0 from a vertex $v \in V$ if every play consistent with σ_0 from v is won by P_0.
- Winning region W_0 of P_0: set of vertices from which P_0 has a winning strategy.

A game is called regular if Win is a (ω-) regular language.

2 Reachability games

Fix $F \subseteq V$. A play is won by P_0 if it reaches F.

Attractors.

$$\begin{array}{rcl} Attr_0^{(0)}(F) & = & F \\ Attr_0^{(n+1)}(F) & = & Attr_0^{(n)}(F) \cup \\ & & \{v \in V_0 \mid post(v) \cap Attr_0^{(n)}(F) \neq \emptyset\} \\ & & \{v \in V_1 \mid post(v) \subseteq Attr_0^{(n)}(F)\} \end{array}$$

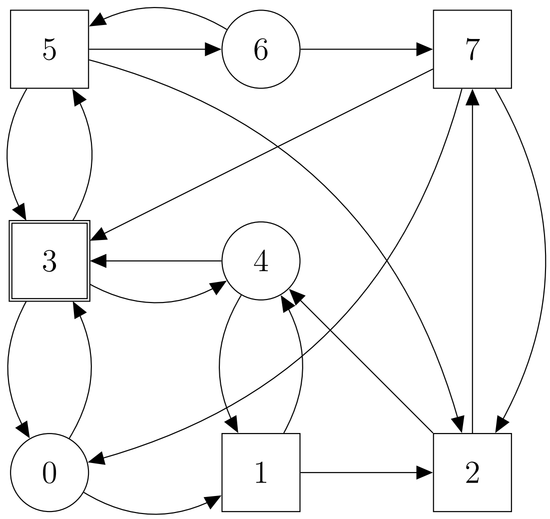
$$Attr_0^{(0)}(F) \subseteq Attr_0^{(1)}(F) \subseteq \cdots \subseteq Attr_0^{(|V|)}(F) =: Attr_0(F)$$

- $Attr_0^{(i)}(F)$ is the set of vertices from which P_0 can reach F in at most i steps.
- The set $Attr_0(F)$ is the winning region of P_0 and its complement $V \setminus Attr_0(F)$ is the winning region of P_1 for the dual condition (*avoid* F).
- The dual of a reachability game is called safety game.
- Reachability and safety games are determined.

The attractor $Attr_1(G)$ of P_1 is defined symmetrically, exchanging P_0 and P_1.

The complement of an attractor of P_0, so a set of the form $U = V \setminus Attr_0(F)$, is a trap for P_0 or (0-trap): P_0 cannot leave U, and P_1 has always a move to stay in U.

1. Compute the winning regions and strategies of P_0 and P_1 for the reachability game below: $F = \{3\}$.



2. Show that on a finite arena, any attractor can be computed in linear time.

3. True or false?

   - $Attr_0(F_1 \cap F_2) = Attr_0(F_1) \cap Attr_0(F_2)$
   - $Attr_0(F_1 \cup F_2) = Attr_0(F_1) \cup Attr_0(F_2)$

4. Assume that we have a game described by a finite game arena $\mathcal{A}$ and a regular, prefix-closed language $L \subseteq V^*$. The winning condition *Win* is the set of all infinite words $w \in V^\omega$ such that all prefixes of w belong to L.

   Give an algorithm to compute the winning regions and strategies of the two players.

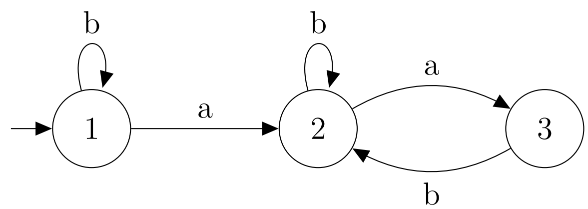
3 ω-automata

An ω-automaton is a tuple $\mathcal{A} = (S, \Sigma, \delta, s_0, Acc)$ with finite set of states S, and transition relation $\delta \subseteq S \times \Sigma \times S$. It accepts words from Σ^ω with various acceptance conditions Acc:

- Büchi: $Acc = F$, and F should be visited infinitely often

- Parity: states have priorities (colors) $p : V \to \{0, \ldots, k\}$. The highest color visited infinitely often should be *even*.

- Rabin/Streett, Muller.
- The automaton $\mathcal{A}$ below accepts with the parity condition (maximal color visited infinitely often is even), with the color of a state being equal to the state number.



  1. Which language is accepted by $\mathcal{A}$?
  2. Give an equivalent Büchi automaton.
  3. Can you give an equivalent *deterministic* Büchi automaton? Justify.

- Show how to transform a Büchi automaton into an equivalent parity automaton with the same number of states.
- Show how to transform a parity automaton into an equivalent Büchi automaton of size polynomial in the size of the original automaton.